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Microwave and Fluctuation Resistance of Superconducting Alloy Films*

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The microwave resistance of dirty superconducting films with magnetic fields just less than the critical values applied parallel to their surfaces has been calculated, together with the paraconductivity due to fluctuations just above the critical fields. Maki's results for the microwave resistance of the surface sheath are corrected and extended to include films of all thicknesses. Singularities similar to those observed in tunneling measurements are predicted to occur at the critical thickness for flux entry. Anisotropies for different relative orientations of the electric and magnetic fields are predicted, which are at a maximum at the flux entry and vanish only in the thin-film limit.

I. INTRODUCTION

Recently, considerable theoretical progress has been made in calculating the dynamic response to electric fields of dirty type-II superconducting alloys in the gapless regime. Schmid[1] and Caroli and Maki[2] (CM) have used the linearized time-dependent Ginzburg-Landau (GL) equation for the order parameter[3] to calculate its response near the upper critical field H_{C2} and then used the static GL equation for the current to calculate the flux-flow resistance. Gor'kov and Eliashberg[4] (GE) have investigated the dynamics of superconductors containing paramagnetic impurities and found the situation generally much more complicated than envisioned in Refs. 1 and 2 due to the appearance of so-called anomalous contributions, which arise when products of both advanced and retarded response functions must be considered. We[5] have shown that such an anomalous term, which was not considered in Refs. 1 and 2, does contribute to the flux-flow current at nonzero temperatures and must be added to their results.

Similar techniques can be employed to calculate the response of superconductors to microwave fields. Such calculations have been made by Maki,[6] valid only for E parallel to the static field H_0, and by Caroli and Maki,[7] illustrating the anisotropy between the two geometrical configurations $E \parallel H_0$ and $E \perp H_0$. Unfortunately, the latter results were also incorrect and did not reduce to the correct zero-frequency limit until certain other corrections, which we found,[5] were included to obtain the complete anisotropy of the penetration depth for static electric fields (no Meissner screening of E when $E \perp H_0$).

The method of CM[7] was applied by Fischer and Maki[8] to the impedance of the surface sheath of very thick films near H_{C3}, and an anisotropy between the two geometrical configurations was noted.[9] However, the deficiency of the calculation of Ref. 7 was continued in Ref. 8. We therefore correct these calculations in Sec. II of this paper using the results of our Ref. 5. The corrected results are simpler than those of Ref. 8, and a temperature-independent value of the anisotropy is obtained. The new result is not so strikingly different from the previous one as in the flux-flow case since the anisotropy is no longer total. The two results differ by less than the experimental errors owing to the unfortunate choice of strong-coupling lead alloy samples, whose properties may differ significantly from predictions of the weak-coupling theories.

In Sec. III we extend these results to films which are thin compared with the rf skin depth but of size comparable with the temperature-dependent coherence length $\xi(t)$. Just as for the tunneling characteristics,[10] we find discontinuities of slope of the response at the critical thickness $d_c = 1.812\xi(t)$ for vortex entry, which suddenly marks the change between the thin-film and surface-sheath regimes. The anisotropy of the response reaches a maximum at d_c and disappears entirely for very thin films.

As previously in Ref. 5, we can easily extend these results for the response of the mean-field superconducting state to find the leading corrections to the normal-state conductivity σ above $T_c(H)$ or $H_c(T)$ due to superconducting fluctuations. Usadel[11] and Maki[12] have already noted that in bulk samples just above H_{C2} the temperature or field dependence of the paraconductivity σ' should be completely different depending on the relative orientations of E to H_0, reflecting the anisotropy to be observed below H_{C2}, where the conductivity is infinite along the vortex lines but finite perpendicular to them. In Sec. IV we calculate σ' for the surface sheath of thick films $d \gg \xi(t)$ and, more generally, for all film thicknesses in a parallel field. The results for $E \parallel$ or $\perp H_0$ have the same temperature dependence but are generally anisotropic. As a function of H_0 we predict a singularity at the flux-entry field $H_{\mathrm{FE}} = 1.623/2ea^2 = 0.812\varphi_0/\pi a^2$, where $a = \frac{1}{2}d$ and

φ_0 is the flux quantum. For very thin films, the isotropic result of Aslamazov and Larkin[13] (AL) in a parallel field is recovered.

II. SURFACE RESISTANCE OF THICK FILM

Let us consider microwaves emitted at $x=-\infty$ striking the surface $x=0$ of a film, which is thick compared with the skin depth. The incoming wave may be characterized by a vector potential $A_y = e^{i\omega(x-t)}$. (We choose units $c=\hbar=k_B=1$.) The impedance is $Z=4\pi E(0)/H_\omega(0)=4\pi i\omega A(0)/\partial_x A(0)$. Inside the film the vector potential obeys $-\partial_x^2 A(x) = 4\pi j(x)$. The current j may be divided into two parts: the normal-state current $j_n = -Q_n A = i\omega\sigma A$ and the additional current j' arising from the change Q' in the response function Q due to superconductivity. We assume a static field H_0 is applied parallel to the film surface with a magnitude near H_{C3} so that j' is a small perturbation and will find the leading correction to Z_n. In terms of the skin depth $\delta=(-4\pi i\omega\sigma)^{-1/2}$ we must solve

$$(-\partial_x^2+\delta^{-2})A(x)=4\pi j'(x) \ . \tag{1}$$

This equation is conveniently solved by introducing a Green's function G:

$$\begin{aligned}(-\partial_x^2+\delta^{-2})G(x,x') &= \delta(x-x') \ , \\ G(x,x') &= \tfrac{1}{2}\delta e^{-|x-x'|/\delta} \ .\end{aligned} \tag{2}$$

The solution $A(x)$ may be written immediately using G:

$$\begin{aligned}A(x) &= A_0 e^{-x/\delta} + \int_0^\infty G(x,x')4\pi j'(x')\,dx' \\ &= A_0 e^{-x/\delta} + \tfrac{1}{2}\delta\int_0^x e^{(-x+x')/\delta}\,4\pi j'(x')\,dx' \\ &\quad + \tfrac{1}{2}\delta\int_x^\infty e^{(x-x')/\delta}4\pi j'(x')\,dx' \ .\end{aligned} \tag{3}$$

The impedance Z may be computed knowing $A(x)$:

$$Z=-4\pi i\omega\delta\,\frac{A_0+2\pi\delta\int_0^\infty e^{-x/\delta}j'(x)\,dx}{A_0-2\pi\delta\int_0^\infty e^{-x/\delta}j'(x)\,dx} \ . \tag{4}$$

To obtain the leading correction to Z, j' is computed in the presence of the normal-state potential $A_0 e^{-x/\delta}$. For example, if Q' is local, then $j'(x) = -Q'(x)A_0 e^{-x/\delta}$, and if further Q' is constant, we get

$$Z=\frac{4\pi i\omega}{(4\pi Q_n)^{1/2}}\left(1-\frac{Q'}{2Q_n}\right) , \tag{5}$$

which is the correct first-order expansion of

$$Z=4\pi i\omega/[4\pi(Q_n+Q')]^{1/2} \ .$$

The response function Q' of the surface sheath is, however, not constant but becomes exponentially small at distances greater than the temperature-dependent coherence length $\xi(t)$ from the surface. To simplify the integrals we consider the case $\xi(t)\ll \mathrm{Re}\,\delta$, which for the "dirty" alloys we are considering requires

$$[\omega/\pi(T_{c0}-T)]^{1/2}\ll\sqrt{2}\,\kappa \ . \tag{6}$$

This condition excludes only a small range of tempertures T near the zero-field critical temperature T_{c0} since we assume the microwave frequency $\omega\ll\pi T_{c0}$, and the GL parameter κ must be $>0.59/\sqrt{2}$ for the transition to be second order.

The previous results for the vortex state[5-7] may be easily extended to calculate Q' for the surface sheath by changing appropriately the average value of the order-parameter squared $\langle|\Delta|^2\rangle$ and the spectrum of excited states and matrix elements. As before, we consider first the simpler geometry with the rf electric field E parallel to the static magnetic field H_0. In this case Q' is a local function of x. The principal contribution to the surface resistance comes from $\mathrm{Re}Q'$, which is typically larger than $\mathrm{Im}Q'$ by a factor of $8\pi T_{c0}/\omega$. From Eqs. (2), (3), and (5) of Ref. 5 we obtain

$$\mathrm{Re}Q'_{\parallel}(x)=\frac{\sigma}{\pi T}\,\psi'(\tfrac{1}{2}+\rho)\,|\Delta(x)|^2\left(1+\frac{1}{2}\,\frac{\omega^2}{\omega^2+\epsilon_0^2}\right) \ . \tag{7}$$

As in Ref. 5, $\rho=\epsilon_0/4\pi T$ is the solution of $\psi(\frac{1}{2}+\rho)-\psi(\frac{1}{2})=\ln(t)$, where $t=T/T_{c0}$. ψ is the digamma function, and ψ' and ψ'' are its first and second derivatives. For the surface sheath, the ground-state energy $\epsilon_0=0.590D2eH_{C3}=D\xi^{-2}(t)$, where D is the diffusion constant. The last term in Eq. (7) is only important for $\kappa\gg1$, since otherwise condition (6) will be violated before $\omega\sim\pi(T_{c0}-T)$. Maki[14] has evaluated $|\Delta|^2$ using the Gaussian approximation:

$$\frac{\sigma}{\pi T}\,\psi'(\tfrac{1}{2}+\rho)\xi^{-1}(t)\int_0^\infty|\Delta(x)|^2\,dx = \left(\frac{\pi}{2}\right)^{1/2}\frac{0.59e}{\pi}\,\frac{H_{C3}(t)-H_0}{2\kappa_2^2(t)-0.311} \ . \tag{8}$$

Inserting Eqs. (7) and (8) into Eq. (4), one gets a result which, when $\omega^2\ll\epsilon_0^2$, is the same as one already obtained by Maki[8]:

$$\begin{aligned}Z_{\parallel} &= -4\pi i\omega\delta\left[1-4\pi\delta\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\pi}\,\frac{H_{C3}(t)-H_0}{\xi(t)H_{C3}(t)}\right. \\ &\quad\left.\times\frac{1}{2\kappa_2^2(t)-0.311}\left(1+\frac{1}{2}\,\frac{\omega^2}{\omega^2+\epsilon_0^2}\right)\right] \ , \\ R_{\parallel} &= \mathrm{Re}Z_{\parallel}=R_n\left[1-(2\pi)^{1/2}\frac{\delta_0}{\xi(t)}\,\frac{H_{C3}(t)-H_0}{H_{C3}(t)}\right. \\ &\quad\left.\times\frac{1}{2\kappa_2^2(t)-0.311}\left(1+\frac{1}{2}\,\frac{\omega^2}{\omega^2+\epsilon_0^2}\right)\right] ,\end{aligned} \tag{9}$$

where $\delta_0^{-1}=\mathrm{Re}\,\delta^{-1}=(2\pi\omega\sigma)^{1/2}$ and $R_n=(2\pi\omega/\sigma)^{1/2}$.

Our numerical calculations show that the corrections to the Gaussian approximation are only a few percent. The factor $(2\pi)^{1/2} = 2.51 \to 2.63$, and the last constant in the denominator $(\sqrt{2}-1)(4-\pi)/(\pi-2) = 0.311 \to 0.328$. Since this leading correction to Z is purely real, corrections to $\chi = \mathrm{Im}Z$ will be smaller than those to R in this limit. The leading correction $\sim \omega/(T_{c0}-T)$ is obtained by replacing $\mathrm{Re}Q'$ by $\mathrm{Im}Q'$ from Eqs. (2) and (5) in Ref. 5, and the correction $\sim \kappa^{-1}[\omega/(T_{c0}-T)]^{1/2}$ by expanding the exponent in Eq. (4) to get a correction factor $-2\langle x\rangle/\delta = -2(0.59)^{1/2}\xi(t)/\delta$. Very near T_{c0}, where the inequality (6) is reversed, Q' is approximately constant over the skin depth, and the leading correction to Z is that given in Eq. (9) multiplied by $4\delta/(\sqrt{\pi})\xi(t)$.

If E is oriented perpendicular to H_0, two additional contributions to Q' arise. The first one $Q'_{\perp 1}$ was the only one considered by Maki. This contribution results from the change of Δ produced by E and is expressed in terms of a sum over the normalized excited states of the surface sheath $|n\rangle$ with n nodes and corresponding eigenvalues ϵ_n. To simplify the algebra we will ignore corrections $\sim \omega/\pi(T_{c0}-T)$ here and assume for the moment $\kappa \gg 1$. Again the main interest is in $\mathrm{Re}Q'$. Referring to CM[7] or to Eqs. (3) and (6) of Ref. 5, we have

$$\mathrm{Re}Q'_{\perp 1} = -16\sigma|\Delta(x)|^2 D \sum_{n>0} [\langle 0|2eH_0(x-x_0)|n\rangle]^2 \times \frac{\psi(\frac{1}{2}+\rho_n)-\psi(\frac{1}{2}+\rho)}{(\epsilon_n-\epsilon_0)^2}, \qquad (10)$$

where $\rho_n = \epsilon_n/4\pi T$ and $\rho_0 = \rho$ as before. In general, $x_0 = \langle 0|x|0\rangle$, which equals $0.590\xi(t)$ for the surface sheath, minimizing ϵ_0. The terms $\langle 0|H_0x_0|n\rangle = H_0x_0\langle 0|n\rangle = 0$ by orthogonality. Near H_{C2} in the vortex state, $x_0 = 0$ and the only nonvanishing matrix element is $\langle 0|2eH_0x|1\rangle = (eH_0)^{1/2}$. Inserting the value $\rho_1 = 3\rho$, the previous results for the vortex state are recovered. For the surface sheath our computer calculations[15] show that the matrix element squared for $n=1$ contains most of the sum:

$$(\langle 0|2eH_0x|1\rangle)^2 = 0.934\langle 0|[2eH_0(x-x_0)]^2|0\rangle = (0.934)(0.590)\,eH_0 .$$

A good approximation, used by Maki,[8] is to replace ϵ_n by $\epsilon_1 = 5.62\epsilon_0$ and then sum over n. Maki thus obtained a temperature-dependent anisotropy coefficient $A_3(t)$:

$$A_3(t) = \frac{R'_\perp}{R'_\parallel} = \frac{\mathrm{Re}\int_0^\infty (Q'_\parallel + Q'_{\perp 1})\,dx}{\mathrm{Re}\int_0^\infty Q'_\parallel\,dx} = 1 - \frac{2[\psi(\frac{1}{2}+5.62\rho)-\psi(\frac{1}{2}+\rho)]}{(4.62)^2\rho\psi'(\frac{1}{2}+\rho)} . \qquad (11)$$

This function varies between 0.57 at T_{c0} and 0.84 at $T=0$. A similar function $A(t)$ found by CM[7] for the vortex state varies between 0 at T_{c0} and 0.45 at $T=0$

As we pointed out earlier,[5] these results of CM for $E\perp H_0$ are wrong, and $A(t)$ must vanish for all T in order for the flux-flow resistance to be finite as observed. Since Maki's calculation is based on CM, a similar conclusion is obtained for the surface sheath, although the results of their error are not so dramatic here since the anisotropy is incomplete, $A\neq 0$. Maki's result, $A_3(t)$, is also only correct at T_{c0}. The additional terms Q'_{12} are obtained by generalizing slightly Eqs. (11) and (12) of Ref. 5:

$$\mathrm{Re}Q'_{\perp 2} = -16\sigma|\Delta(x)|^2 D\sum_{n>0} [\langle 0|2eH_0(x-x_0)|n\rangle]^2 \times\left[\frac{\psi'(\frac{1}{2}+\rho)}{4\pi T(\epsilon_n-\epsilon_0)} - \frac{\psi(\frac{1}{2}+\rho_n)-\psi(\frac{1}{2}+\rho)}{(\epsilon_n-\epsilon_0)^2}\right] . \qquad (12)$$

Just as in the vortex state, there is considerable cancellation between $Q'_{\perp 1}$ and $Q'_{\perp 2}$, and a simpler temperature-independent result for the anisotropy coefficient is obtained:

$$A = 1 - \sum_{n>0} \frac{4D(\langle 0|2eH_0x|n\rangle)^2}{\epsilon_n - \epsilon_0} . \qquad (13)$$

For the vortex state the matrix element squared is eH_0, $\epsilon_1 = 3\epsilon_0$, $2eDH_{C2} = \epsilon_0$, and therefore $A=0$. For the surface sheath, Maki's approximation gives $A = 0.57$, and actually summing over n by computer, we get $A = 0.586$.

For smaller values of $\kappa\sim 1$ there are further corrections to $Q'_{\perp 1}$ due to the finite value of the skin depth. Referring to Eq. (10), the matrix element of the product of the rf and static fields is actually $\langle 0|2eH_0(x-x_0)\,e^{-x/\delta}|n\rangle$. For large $\kappa \gg 1$ we could neglect terms $\sim x/\delta$ and replace the exponential by 1, so that the $n=0$ matrix element vanished. Keeping the first term in the expansion $\sim -x/\delta \sim -\xi(t)/\delta \ll 1$ results in a finite value for the $n=0$ matrix element, which is important because the denominator of Eq. (10) vanishes as one higher power of $\epsilon_n - \epsilon_0$ than the numerator. [If we check Eq. (11), we find no such problem arises for $\mathrm{Re}Q'_{\perp 2}$.] Hence the value of the $n=0$ term is considerably enhanced, and we must look for small corrections to the denominator. Referring to CM,[7] the extra denominator factor is the inverse of the time-dependent GL equation $[-i\omega + \delta\epsilon + O(|\Delta|^2)]^{-1}$. When $\delta\epsilon = \epsilon_n - \epsilon_0$ vanishes, these extra terms become important. There are two types of terms $\sim |\Delta|^2$. The anomalous term of Eliashberg[4] is

$$\sim \frac{-i\omega}{-i\omega + D\nabla^2}\,\frac{|\Delta|^2}{\epsilon_0} \sim \frac{-i\omega|\Delta|^2}{\epsilon_0^2} .$$

The validity of the theory, as in the vortex case, is restricted to the gapless region where $|\Delta| < \epsilon_0$, where we can ignore this term in comparison with the $-i\omega$ term. The second type of term $\sim |\Delta|^2$ is the one which appears in the usual static GL equation, whose standard evaluation has been referred to and used above in Eq. (8). The matrix element squared $[\langle 0|2eH_0(x-x_0)x/\delta|0\rangle]^2$ is easily evaluated to be $(0.590)^2/4\delta^2 = -\pi i\omega\sigma(0.590)^2$. Taking the limit of Q'_{11} [Eq. (10)] as $\epsilon_n \to \epsilon_0$, we obtain the additional correction Q'_{13}:

$$\int_0^\infty Q'_{13}\,dx = -\frac{4D\sigma^2}{T}(0.590)^2 \int_0^\infty |\Delta|^2\,dx\,\psi'(\tfrac{1}{2}+\rho) \times \frac{-i\omega}{-i\omega + 4eD[H_{C3}(t)-H_0]} . \quad (14)$$

Taking the real part of Eq. (14), we obtain the corrected value of the anisotropy coefficient A:

$$A = 0.586 - \frac{(0.590)^2}{4(1.20\kappa)^2}\,\frac{\omega^2}{\omega^2 + \{2\epsilon_0[H_{C3}(t)-H_0]/H_{C3}(t)\}^2} . \quad (15)$$

III. DYNAMIC RESPONSE OF THINNER FILM

In order to study the response of a film in the interesting range of thicknesses $d \sim \xi(t)$ we now assume $d \ll \delta_0$, which is the opposite of the assumption made in Sec. II. We again assume an incoming wave in a vacuum $A_y = e^{i\omega(x-t)}$. An infinitely long and wide film lies between $x = \pm a = \pm\frac{1}{2}d$. First we consider the case where the film is backed $(x > a)$ by a thick insulating substrate with index of refraction n. (The problem of a substrate with thickness approximately equal to the microwave length and the resulting resonances can be easily handled at the expense of somewhat more cumbersome algebra.[16]) Using the Green's function (2) to calculate the leading correction to the response satisfying boundary conditions at $x = \pm a$, we find the magnitude of the reflected wave A_r and of the transmitted wave A_t:

$$A_r = \frac{1-n-8\pi\sigma a - (i\omega)^{-1}4\pi\int_{-a}^{a} j'(x)\,dx}{1+n+8\pi\sigma a} ,$$
$$A_t = \frac{2-(i\omega)^{-1}4\pi\int_{-a}^{a} j'(x)\,dx}{1+n+8\pi\sigma a} . \quad (16)$$

The value of A inside the film is to lowest order the same as A_t. Thus, in terms of the average value of the change of the film conductivity $\sigma' = \int_{-a}^{a} Q'(x)\,dx/(-i\omega 2a)$, we can evaluate

$$(i\omega)^{-1}4\pi\int_{-a}^{a} j'(x)\,dx = 16\pi a\sigma'/(1+n+8\pi\sigma a).$$

Thus we see that in the limit $d \ll \delta_0$ only the average value σ' enters and, as expected, appears exactly as the expansion of the combination $\sigma + \sigma'$:

$$A_r = \frac{1-n-8\pi\sigma a}{1+n+8\pi\sigma a} - \frac{16\pi\sigma' a}{(1+n+8\pi\sigma a)^2}$$
$$= \text{expansion of } \frac{1-n-8\pi(\sigma+\sigma')a}{1+n+8\pi(\sigma+\sigma')a} ,$$
$$A_t = \frac{2}{1+n+8\pi\sigma a} - \frac{16\pi\sigma' a}{(1+n+8\pi\sigma a)^2} \quad (17)$$
$$= \text{expansion of } \frac{2}{1+n+8\pi(\sigma+\sigma')a} .$$

The leading corrections to the reflection coefficient $|A_r|^2$ and the transmission coefficient $|A_t|^2$ only involve Reσ'. The phase shift is proportional to Imσ'. In practical units, $8\pi\sigma a = Z_0/R_\square$, where $Z_0 = 376.6\ \Omega$ and $R_\square$ is the resistance per square area of the film.

If the thin film is backed by a thin insulator (to eliminate proximity effects) and then by a thick normal metal (such as the wall of a microwave cavity) instead of by a thick insulator, the thin film will play the same role the surface sheath played in Sec. II. From Eq. (4) we obtain the impedance Z including the leading correction due to σ', the difference between the average conductivity of the film and of the backing metal: $Z = -4\pi i\omega\delta(1-\sigma' d/\sigma\delta)$. As before, the leading correction to the surface resistance ReZ' involves only Imσ' and not the (real) difference in the normal-state conductivities. Thus by varying the backing medium both Reσ' and Imσ' may be measured by absorption experiments.

We will calculate σ' when a static field H_0 is applied parallel to the film with a magnitude near the critical value $H_{C\parallel}(t)$, so that the film is gapless. Then for $E \parallel H_0$ we can again use Eqs. (2), (3), and (5) of Ref. 5:

$$\text{Re}\sigma'_\parallel = \frac{-\text{Im}Q'_\parallel}{\omega} = L\left(\frac{\epsilon_0^2}{\omega^2+\epsilon_0^2} + 3\rho\frac{\psi''(\frac{1}{2}+\rho)}{\psi'(\frac{1}{2}+\rho)}\right) ,$$
$$\text{Im}\sigma'_\parallel = \frac{\text{Re}Q'_\parallel}{\omega} = L\frac{2\epsilon_0}{\omega}\left(1+\frac{1}{2}\frac{\omega^2}{\omega^2+\epsilon_0^2}\right), \quad (18)$$

where L is a common amplitude coefficient,

$$L = \frac{\sigma}{2\pi T\epsilon_0}\langle|\Delta|^2\rangle\,\psi'(\tfrac{1}{2}+\rho). \quad (19)$$

The spatially averaged magnitude of the order parameter squared $\langle|\Delta|^2\rangle$ may be evaluated numerically for any film thickness using the procedure developed in Ref. 10. For the tunneling experiments in Ref. 10 we were interested in calculating the order parameter at a boundary $|\Delta(\pm a)|^2$. We obtained this value in terms of functions J_1 and J_2 defined and illustrated in Figs. 9 and 10 there:

$$|\Delta(\pm a)|^2 = \frac{8\pi TD}{\psi'(\frac{1}{2}+\rho)}\,\frac{(1.20\kappa)^2 J_1}{\kappa_2^2 - J_2}\,2e(H_{C\parallel}-H_0) . \quad (20)$$

J_1 and J_2 are functions of $\epsilon = [a/\xi(t)]^2 = a^2\epsilon_0/D$, while κ_2 is a function of both ϵ and t. Using Eq. (20), we obtain

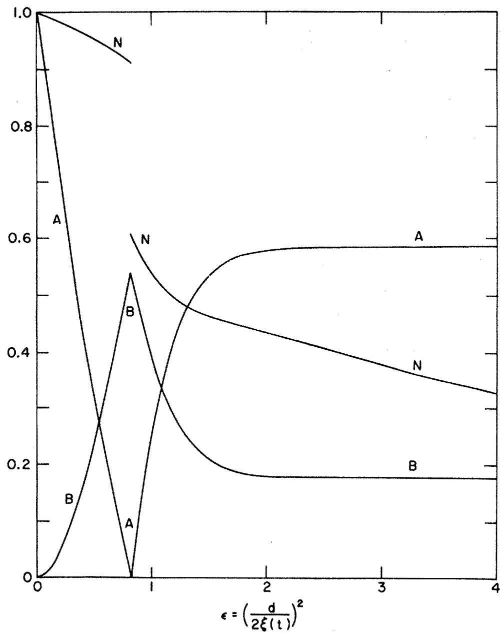

FIG. 1. Three functions N, A, and B necessary for calculating the microwave conductivity and its anisotropy.

$$L = \frac{4D\sigma}{\epsilon_0}\,\frac{\langle|\Delta|^2\rangle}{|\Delta(\pm a)|^2}\,J_1\,\frac{(1.20\kappa)^2}{\kappa_2^2 - J_2}\,2e(H_{C\parallel} - H_0)\,. \quad (21)$$

It is useful to compute a function $N(\epsilon)$:

$$N = \frac{h}{\epsilon}\,\frac{\langle|\Delta|^2\rangle}{|\Delta(\pm a)|^2}\,J_1, \quad (22)$$

where the dimensionless quantity $h = 2eH_{C\parallel}a^2$. Then

$$L = 4\sigma N\,\frac{(1.20\kappa)^2}{\kappa_2^2 - J_2}\,\frac{H_{C\parallel} - H_0}{H_{C\parallel}}\,. \quad (23)$$

The results of the computer calculation for N (along with two functions A and B which are important for $E \perp H_0$) are shown in Fig. 1. The most interesting feature is the discontinuity of $\frac{3}{2}$ at the critical thickness, which was verified in the tunneling measurements. We have improved our numerical calculations slightly recently, so in Table I we summarize the interesting limiting values of the previous functions J_1, J_2, and C (although the changes are rather insignificant, always < 1%) along with the new functions N, A, and B.

The leading corrections to our formulas (17) for

TABLE I. Limiting values of six functions of $\epsilon = [a/\xi(t)]^2$.

	$\epsilon \to 0$	$\epsilon = \epsilon_{cr} = 0.821$	$\epsilon \gtrsim 4$	Gaussian approx. for $\epsilon \to \infty$ (A and B using ϵ_1)
J_1	$(\frac{1}{3}\epsilon)^{1/2}$	0.361, 0.241	0.385	0.42
J_2	0.80ϵ	0.535	0.164	0.156
C	0.30ϵ	0.260	0.877	1.00
N	1	0.912, 0.608	$0.657\epsilon^{-1/2}$	$0.63\epsilon^{-1/2}$
A	$1-1.60\epsilon$	0	0.586	0.57
B	$1.30\epsilon^2$	0.541	0.177	0.187

the reflected and transmitted waves A_r and A_t are obtained by taking the next term in the expansion of $\tanh(a/\delta)$. Each factor

$$4\pi\sigma a \to \frac{1}{i\omega\delta}\left[\frac{a}{\delta} - \frac{1}{3}\left(\frac{a}{\delta}\right)^3\right] = 4\pi\sigma a\,(1 + \tfrac{1}{3}\,i\omega\,4\pi\sigma a^2).$$

[Those corrections falling under integral signs for σ' will be smaller if $a > \xi(t)$ since a^2 will be replaced by $\xi^2(t)$.] The most important effect of these corrections will be on the magnitudes $|A_r|$ and $|A_t|$ because the correction factor is imaginary and thus allows $\mathrm{Im}\sigma'$, which is generally $\gg \mathrm{Re}\sigma'$, to contribute. Relative to the previous effects proportional to $\mathrm{Re}\sigma'$ this correction

$$\sim \frac{1}{3}\,\omega 4\pi\sigma a^2\,\frac{\mathrm{Im}\sigma'}{\mathrm{Re}\sigma'} \sim \frac{2}{3}\,\epsilon_0 4\pi\sigma a^2 \sim \frac{2}{3}\,\frac{4\pi\sigma D}{\xi^2(t)}\,a^2 \sim \frac{1}{6}\,\frac{1}{(1.20\kappa)^2}\,\frac{a^2}{\xi^2(t)}\,.$$

Recognizing that the temperature-dependent static penetration depth $\lambda(t) = \kappa\xi(t)$, the corrections $\sim 0.1a^2/\lambda^2(t)$ and result from a mixing between the static and dynamic screening. Generally these corrections will be very small in the interesting region $a \sim \xi(t)$, since κ^2 must be > 0.535 for the transition to be of second order and the theory to apply at all. Consequently we will not calculate these screening corrections in detail but merely notice that they could be important for thick films before the condition $a \sim \delta_0$ is obtained, since $\lambda(t) < \delta_0$ when $\omega < \pi\,(T_{c0} - T)$.

Turning to the other geometrical arrangement $E \perp H_0$, we can obtain the additional contributions $Q'_{\perp 1}$ and $Q'_{\perp 2}$ by generalizing Eqs. (3), (6), and (11) of Ref. 5:

$$\langle Q_{\perp 1}\rangle = -16\sigma\langle|\Delta|^2\rangle D \sum_{n>0} (\langle 0|\,2eH_0 x\,|n\rangle)^2 \times \left[\frac{\psi(\frac{1}{2}+\rho_n) - \psi(\frac{1}{2}+\rho)}{(\epsilon_n - \epsilon_0)^2} - \frac{i\omega}{4\pi T}\left(\frac{\psi'(\frac{1}{2}+\rho_n) - 2\psi'(\frac{1}{2}+\rho)}{(\epsilon_n - \epsilon_0)^2}\right)\right]\,,$$

$$\langle Q_{\perp 2}\rangle = -16\sigma\langle|\Delta|^2\rangle D\sum_{n>0}(\langle 0|2eH_0x|n\rangle)^2 \times\left[\frac{\psi'(\frac{1}{2}+\rho)}{4\pi T(\epsilon_n-\epsilon_0)}-\frac{\psi(\frac{1}{2}+\rho_n)-\psi(\frac{1}{2}+\rho)}{(\epsilon_n-\epsilon_0)^2}-\frac{i\omega}{4\pi T}\left(\frac{\psi''(\frac{1}{2}+\rho)}{4\pi T(\epsilon_n-\epsilon_0)}-\frac{\psi'(\frac{1}{2}+\rho_n)-\psi'(\frac{1}{2}+\rho)}{(\epsilon_n-\epsilon_0)^2}\right)\right] . \quad (24)$$

The expansion here is in powers of $\omega/(\epsilon_1-\epsilon_0)$, which does not diverge at T_{c0} since $\lim\epsilon_1 = D\pi^2/4a^2$ as $\epsilon_0\to 0$. Thus

$$\lim_{T\to T_c}\frac{\omega}{\epsilon_1-\epsilon_0}=\frac{\omega}{\pi T_{c0}}\frac{a^2}{2\xi^2} ,$$

where $\xi^2(t)=\xi^2/(1-t)$ near T_{c0}. In terms of $T_{\rm FE}$, the temperature at which the flux entry occurs at the critical value $H_{C\parallel}$ $\omega/\epsilon_1\to 0.4\omega/\pi(T_{c0}-T_{\rm FE})$. Thus, if the film is not too thick, so that $T_{\rm FE}$ is not too close to the limit as $T\to T_c$, we can ignore these finite frequency corrections, since we assumed $\omega\ll\pi T_{c0}$. As above in this section, we will not calculate in detail here the corrections $\sim\kappa$ including those of the $Q'_{\perp 3}$ type of Sec. II.

Again considerable cancellation occurs in the sum $Q'_{\perp 1}+Q'_{\perp 2}$, and only ψ functions of the ground-state value ρ survive, so that the sums over eigenvalues will depend only on the thickness ratio $\epsilon=[a/\xi(t)]^2$ and not also explicitly on temperature. Adding these results to our previous results for $\sigma'_\parallel$, Eq. (18), we obtain $\sigma'_\perp$:

$$\mathrm{Re}\sigma'_\perp = L\left[\frac{\epsilon_0^2}{\omega^2+\epsilon_0^2}+B+(1+2A)\rho\frac{\psi''(\frac{1}{2}+\rho)}{\psi'(\frac{1}{2}+\rho)}\right] ,$$
$$\mathrm{Im}\sigma'_\perp = L\frac{2\epsilon_0}{\omega}\left(A+\frac{1}{2}\frac{\omega^2}{\omega^2+\epsilon_0^2}\right) , \quad (25)$$

where L is given by Eq. (23), A by Eq. (13), and

$$B=8\epsilon_0 D\sum_{n>0}\frac{(\langle 0|2eH_0x|n\rangle)^2}{(\epsilon_n-\epsilon_0)^2} . \quad (26)$$

The frequency-dependent corrections $\sim\omega^2/\epsilon_0^2$ only become important as $\epsilon_0\to 0$, where the functions A and B take their limiting values and the anisotropy vanishes for thin films if we assume as above that $\omega\ll 2\pi(T_{c0}-T_{\rm FE})=2\pi T_{c0}\xi^2/a^2$. Thus the anisotropy of $\mathrm{Im}\sigma'$ is just given by the function $A(\epsilon)$ as illustrated in Fig. 1. Complete anisotropy is obtained at the flux-entry field $H_{\rm FE}$, just as for the bulk vortex state.

The anisotropy of $\mathrm{Re}\sigma'$ is slightly more complicated:

$$\frac{\mathrm{Re}\sigma'_\perp}{\mathrm{Re}\sigma'_\parallel}=\frac{1+B+(1+2A)\rho\psi''(\frac{1}{2}+\rho)/\psi'(\frac{1}{2}+\rho)}{1+3\rho\psi''(\frac{1}{2}+\rho)/\psi'(\frac{1}{2}+\rho)} . \quad (27)$$

The function $\rho\psi''(\frac{1}{2}+\rho)/\psi'(\frac{1}{2}+\rho)$ varies smoothly between -1 at $T=0$ and 0 at T_{c0}. A plot of it plus 2 is given in Fig. 1 of Ref. 2. The combination occurring in the denominator is plotted in Fig. 2 of Ref. 6. The denominator, $\mathrm{Re}\sigma'_\parallel$, is predicted to vanish at $t=0.6$ and to be negative for lower temperatures. If ϵ is large (>2) at low temperatures, $\mathrm{Re}\sigma'_\perp$ will vanish at $t=0.4$. $\mathrm{Re}\sigma'_\perp$ will always be positive at $\epsilon_{\rm cr}$ and have a peak with a discontinuity of slope. If ϵ is between 0.64 and 0.96 at $T=0$, $\mathrm{Re}\sigma'_\perp$ will remain positive at all temperatures. Outside this range it will change signs only once and be negative at $T=0$.

IV. FLUCTUATION CONDUCTIVITY OF FILMS IN PARALLEL FIELDS

Anisotropies like those derived in Secs. II and III for the leading changes in the conductivity of films just below the critical field $H_{C\parallel}(t)$ are also found in the paraconductivity due to fluctuations above $H_{C\parallel}(t)$. Usadel[11] and Maki[12] have already shown that a large anisotropy, and in fact different power laws, are obtained in bulk superconductors just above $H_{C2}(t)$. These references considered only the regular AL contributions to σ'. (Maki[12] made reference to his previous calculation[17] of the anomalous contributions but did not explicitly state that the two contributions should be summed.) They found the paraconductivity much greater for $E\parallel H_0$, diverging as $T-T_c$ or $H-H_c$ to the power $-\frac{3}{2}$, whereas for $E\perp H_0$ the power is $-\frac{1}{2}$. [Inclusion of the anomalous terms results in the multiplication of σ' by the function[5,7] $L_D(t)$ when $E\perp H_0$.]

We will here consider the most important fluctuations just above $H_{C\parallel}(t)$, which will be in the surface sheath for thick films. Near $H_{C\parallel}(t)$ the anomalous contributions to σ' are suppressed by the field as for thin films,[5] and we will recover the AL result for thin films. We will consider explicitly only fields which produce small shifts of T_c and which are very near the transition $H_0-H_{C\parallel}(t)\ll H_{C\parallel}(t)$. Results for large shifts of T_c are obtained simply by multiplying by the factor $\psi'(\frac{1}{2}+\rho)/[1-\rho\psi'(\frac{1}{2}+\rho)]\times\psi'(\frac{1}{2})$ derived by Fulde and Maki.[18]

We assume a field H_0 is directed along the z axis and take the vector potential $A_y=H_0x$. The lowest eigenvalue ϵ_0 of the GL equation is found with an order parameter $\Delta(x,y)=F(x)e^{iky}$:

$$D[-\partial_x^2+(2eH_0x-k_0)^2]F(x)=\epsilon_0 F(x) ,$$
$$\partial_x F(x)|_{x=\pm a}=0 . \quad (28)$$

The value of k_0 is determined by minimization of ϵ_0, which implies $\partial\epsilon_0/\partial k=0$, $\partial^2\epsilon_0/\partial k^2>0$. If we consider giving additional spatial dependence to Δ of the form $e^{i(\delta k)y+iqz}$, the minimum eigenvalue increases as $\epsilon_0 + Dq^2 + \frac{1}{2}(\delta k)^2 \partial^2\epsilon_0/\partial k^2$. We could also consider solutions for Δ with higher eigenvalues ϵ_n $(n>0)$, $F(x)$ having n nodes, but fluctuations with such forms will not make important contributions near the transition because of the finite separation of eigenvalues. The function $\frac{1}{2}\partial^2\epsilon_0/\partial k^2$ is evaluated by perturbation theory to be just equal to $DA(\epsilon_0)$, where A is given by Eq. (13). A discussion by Saint-James[19] then clarifies why A must vanish at ϵ_{cr}: For thin films $(\epsilon < \epsilon_{cr})$, $F(x)$ is a symmetric function of x, and $k_0 = 0$. In order for the symmetric solutions to become unstable $\partial^2\epsilon_0/\partial k^2$ must approach 0, which occurs at ϵ_{cr}, after which k_0 rises very rapidly from zero with increasing ϵ.

If the electric field E is oriented parallel to H_0, the coupling with the fluctuation is qE as for AL. The only difference from the AL expression for σ' will be the inclusion of the factor $(2D)^{-1}\partial^2\epsilon_0/\partial k^2 = A$ multiplying $(\delta k)^2$ in the fluctuation propagator and the T_c shift $\delta = (T_{c0} - T_c)/T_c = \pi\epsilon_0/8T_c$ added to $\tau = (T - T_{c0})/T_c$:

$$\sigma'_{\parallel} = \frac{e^2}{4\pi d}\int_{-\infty}^{\infty} \frac{\xi^4 q^2 dq\, d(\delta k)}{\{\tau + \delta + \xi^2[q^2 + A(\delta k)^2]\}^3} \; . \tag{29}$$

The integral reduces to the usual AL form when the scale of k is changed, $k \to k/\sqrt{A}$:

$$\sigma'_{\parallel} = \frac{\sigma\tau_0}{(\sqrt{A})(\tau+\delta)} \; . \tag{30}$$

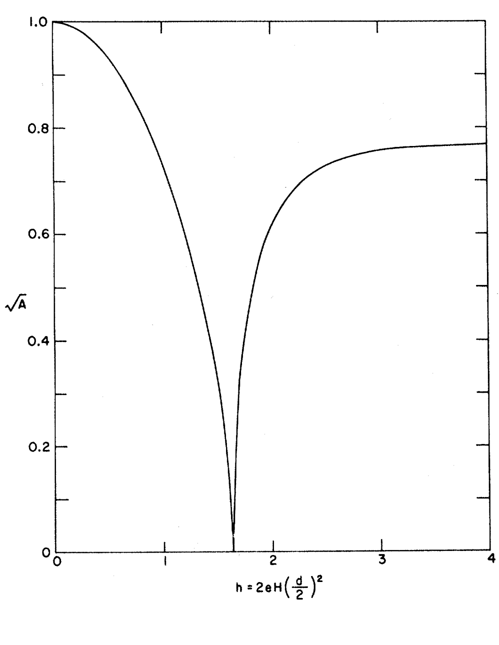

FIG. 2. Square root of the anisotropy function A plotted vs magnetic field.

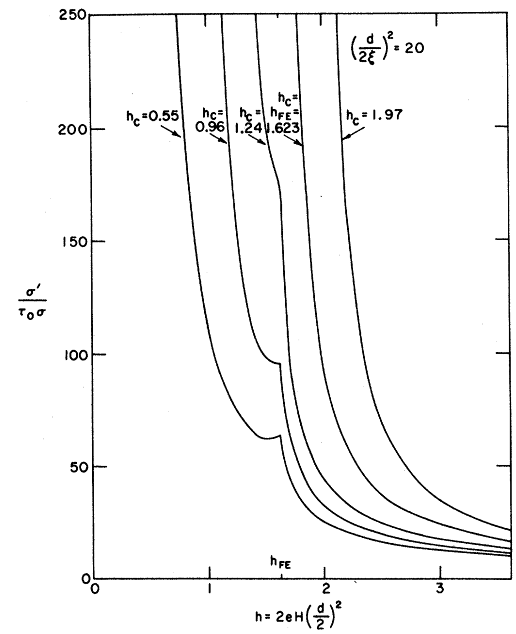

FIG. 3. Paraconductivity for $E \parallel H_0$ presented as a function of field for different temperatures (different values of the critical field h_c). A peak may be seen at the flux-entry field h_{FE}.

If $E \perp H_0$, the coupling is $(2eH_0x - k)E$. If $k = k_0$, $\langle 2eH_0x - k\rangle = 0$ in the ground state, and no net current flows. However, if $k \neq k_0$, $\langle 2eH_0x - k\rangle = -(\delta k)A$:

$$\sigma'_{\perp} = \frac{e^2}{4\pi d}\int_{-\infty}^{\infty} \frac{\xi^4 A^2(\delta k)^2 dq\, d(\delta k)}{\{\tau+\delta+\xi^2[q^2 + A(\delta k)^2]\}^3} \; ,$$

$$\sigma'_{\perp} = \frac{\sigma\tau_0\sqrt{A}}{\tau+\delta} \; . \tag{31}$$

Since $\delta = -\tau$ only right at the transition, it is more convenient here to recognize that δ and A are actually functions of H and a and not of the temperature. Plots of $\epsilon = \delta a^2/\xi^2$ vs $h = 2eH_0a^2$ are presented in several works of Saint-James[19,20] as well as elsewhere.[10] For thick films $\delta = (1.69)(2eH_0\xi^2)$, and for thin films $\delta = \frac{1}{3}(2eH_0a^2)^2$. For convenience we have replotted the function A from Fig. 1 as $\sqrt{A}$ vs h in Fig. 2.

For very thin films $A \to 1$, there is no anisotropy, and the original AL result is recovered. For very thick films near H_{C3} the conductivity of the surface sheath is anisotropic in the same ratio as the microwave surface conductivity: $\sigma'_{\perp}/\sigma'_{\parallel} = A(\infty) = 0.586$.

Perhaps the most interesting case, however, is near the flux-entry field $h_{FE} = 1.623$ for films with ϵ near ϵ_{cr}, since the anisotropy is maximal there. If one holds H fixed and varies T near the transition, the anisotropy should not change. However, if T is fixed and H varied, $\sigma'_\perp$ should peak and $\sigma'_\parallel$ should dip at h_{FE}.

Of course, all contributions to $\sigma'_\parallel$ will not vanish at h_{FE} along with this one, which is usually dominant. In fact, the anomalous contribution to σ', which is the same for $\parallel$ and $\perp$, will peak:

$$\sigma'_a = \frac{2\tau_0}{(\sqrt{A})\tau} \ln\left(\frac{\tau+\delta}{\delta}\right) . \tag{32}$$

Furthermore, the peaks will not be infinite when corrections $\sim \xi^4(\delta k)^4$ are added to the fluctuation propagators. The coefficient of these corrections will be of order unity and will not vanish at h_{FE} like the coefficient of the $(\delta k)^2$ term. These corrections effectively result in a cutoff in the integral over δk in Eq. (29) when $\xi^4(\delta k)^4 \approx \tau + \delta$. Consequently we obtain a more realistic result for $\sigma'_\parallel$ as $A \to 0$:

$$\sigma'_\parallel = \frac{\sigma\tau_0}{(\tau+\delta)[(\tau+\delta)^{1/2}+A]^{1/2}} . \tag{33}$$

A plot of this result is given in Fig. 3 as a function of field for several different temperatures for the particular choice of film thickness $a^2 = 20\xi^2$. One can see from the curves that experimentally the peaks could be smeared out completely by variations of 10% in the film thickness.

V. CONCLUSION

We have calculated the first corrections to the normal-state conductivities of films in parallel magnetic fields above and below the critical-field values. We have predicted interesting anisotropies and singularities which we hope will be investigated further experimentally.

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